ASYMPTOTIC EXPANSIONS OF NAVIER-STOKES SOLUTIONS IN THREE-DIMENSIONS FOR LARGE DISTANCES

Thesis by

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In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California
1961

ACKNOWLEDGMENTS

The author is greatly indebted to Dr. P. A. Lagerstrom, who suggested the present topic, for his guidance and interest during the course of the research. Dr. S. Kaplun also greatly aided this work through several fruitful discussions on special points. Thanks are due also to Mrs. Alrae Tingley for her patient and capable typing of the manuscript.

The author acknowledges with special appreciation the support of the Rand Corporation during the academic year 1960-61.

ABSTRACT

This thesis studies the stationary flow field at large distances from a finite obstacle moving uniformly in a viscous, incompressible fluid. The principal results consist of asymptotic expansions, uniformly valid for large distance, of the velocity and the pressure of the flow field.

The expansion procedure employed is based upon the introduction of a small, extraneous parameter; the construction is thus recast as a perturbation for small values of the parameter. Owing to the presence of a viscous wake, the perturbation is in general a singular one, and is treated accordingly, using methods developed for related hydrodynamical problems.

The calculated results include the following: for the case of axially-symmetric flow, a uniformly valid expansion of the velocity to order r^{-2} inclusive, and of the pressure to order r^{-3} inclusive, r being the distance from the obstacle; for the general case, an expansion of the velocity to order $r^{-3/2}$ and of the pressure to order r^{-2}, inclusive.

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I. INTRODUCTION

Problems related to expansions of Navier-Stokes solutions for large distance have been discussed by a number of authors, especially by Imai (1) and by Chang (2). In particular, it has been shown in reference 2 that the basic problem of construction may be treated by hydrodynamical expansion procedures of the type discussed and illustrated by Lagerstrom, Cole, and Kaplun (3, 4). In the present thesis, the methods of references 2, 3, and 4 are applied to axially-symmetric and strictly three-dimensional Navier-Stokes solutions. Expansion procedures for the three-dimensional cases are discussed and several terms of the expansion are given. It is pointed out also that the procedures are slightly different for the case of three (or more) dimensions, due to certain changes in the nature of cross flow and in the role of the pressure (see Chapter IV, Section 4.1). The axially-symmetric case, regarded as a problem in two dimensions, is discussed separately (Chapter III).

A certain class of Navier-Stokes solutions shall be studied. The basic problem in mind is that of a stationary viscous incompressible flow past a finite three-dimensional solid which tends to a uniform stream at large distance and satisfies the no-slip condition at the solid. Assuming that such a solution is given, we are interested in an asymptotic expansion of the solution for large distance at fixed Reynolds number, more precisely, in an asymptotic expansion valid to all orders r^{-n} as $r \to \infty$, where r is the distance from the origin. The problem studied here, however, will be of a slightly different nature. In the first place, the class of Navier-Stokes solutions studied will be somewhat larger: Given an asymptotic series, it is difficult to determine whether the related Navier-Stokes solutions contain a "solid," i. e. a closed streamsurface. On the other hand, a certain class of Navier-Stokes solutions is related to our series. In the second place, in the present thesis we shall be concerned exclusively with the problem of construction of the series, which, of course, is only a part of the complete problem of asymptotic equality. The special nature of the relationship between our series and the class of Navier-Stokes solutions studied (be it e. g. that of actual asymptotic equality or even that of total equality) is then immaterial for our purposes. On the other hand, a statement of the intended validity of our results is desirable. The class of Navier-Stokes solutions studied and the sense in which our results are intended to be valid are described in Chapter II (see Sections 2. 1, 2. 2).

The methods of reference 2 will be used: An extraneous non-dimensional parameter ϵ (also called the "artificial parameter") is introduced into the exact solution, in such a manner that the expansion for large r may be replaced by a parameter-type expansion for small ϵ. In the present problem, ϵ may be regarded as the ratio of a characteristic length to the length of an extraneous standard of measurement. An "outer" and "inner" expansion are then constructed, representing respectively the repeated applications of an "outer" and an "inner" limit process. The outer expansion is valid for large distance exclusive of the wake, while the inner expansion is valid in the wake. The regions of validity of the two expansions overlap in the sense of reference 4. A "composite" expansion, uniformly valid for large distance may then be constructed from the two principal expansions (see Sections 3.1, 4.2). An advantage of the parametric procedure is that one is first led to approximate partial differential equations, of considerable intuitive importance, while coordinate-type procedures would lead directly to ordinary differential equations.

A number of shortcuts will be used in the course of the construction. However, the construction procedures are explained in Chapter II, where references are also given concerning points which require more elaborate discussion. In particular, reference will be made to two principles: (i) the principle of eliminability and (ii) the principle of transcendental decay of vorticity. The two principles are discussed in Sections 2.3 and 2.4.

II. THE EXACT SOLUTIONS AND THE EXPANSION PROCEDURES

2.1. The Exact Solutions

We consider stationary flows of a viscous incompressible fluid in three dimensions. The following notation is used:

$\vec{q}$ = velocity; p = pressure; x_i = Cartesian coordinates,

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad r^2 = \sum_1^3 (x_i)^2 ; \tag{2-1}$$

ρ = density = constant; ν = kinematic viscosity

The governing equations are the Navier-Stokes equations:

$$(\vec{q} \cdot \nabla)\vec{q} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{q} \tag{2-2a}$$

$$\nabla \cdot \vec{q} = 0 \tag{2-2b}$$

Without loss of generality, we pass directly to the non-dimensional form

$$(\vec{q}^* \cdot \nabla^*)\vec{q}^* = -\nabla^* p^* + \frac{1}{Re}\nabla^{*2}\vec{q}^*, \quad (0 < Re < \infty) \tag{2-3a}$$

$$\nabla^* \cdot \vec{q}^* = 0 \tag{2-3b}$$

$$\nabla^* = \left(\frac{\partial}{\partial x_i^*}\right) \tag{2-3c}$$

of the Navier-Stokes equations. The transformation

$$\vec{q} = U\vec{q}^* \tag{2-4a}$$

$$p = \rho U^2 p^* + P \tag{2-4b}$$

$$x_i = L x_i^* \tag{2-4c}$$

$$Re = \frac{UL}{\nu} \tag{2-4d}$$

sends every solution, $(\vec{q}^*, p^*)$, of equations 2-3 into a family of solutions of equations 2-2 which depends on the dimensional parameters U, L, P, ρ, ν; and, conversely, every solution of equations 2-2 may be so obtained. The question of existence of a characteristic length for a given solution of equations 2-2 is thus expelled from our considerations. We shall consider solutions of equations 2-3 which satisfy the following conditions:

α) There exists a sphere S such that $\vec{q}^*$ and p^* are regular outside S and continuous at infinity.

β) At infinity,

$$\vec{q}^* = \vec{i}, \; p^* = 0 \tag{2-5a}$$

We shall also require

γ) $$\oint\!\!\oint \vec{q}^* \cdot d\vec{s}^* = 0 \tag{2-5b}$$

Condition (γ) is not essential, but leads to a number of well-known dynamical relations concerning flow at large distance. Above, solutions of equations 2-3 are regarded as distinct for distinct values of the Reynolds number, Re. Hence a solution, $(\vec{q}^*, p^*)$, of equations 2-3 is a function of the x_i^* only. Expansions will be constructed for $r^* \to \infty$.

2. 2. Limits and Expansions

Given a solution $(\vec{q}^*, p^*)$ one may introduce an extraneous parameter ϵ and new independent variables $\tilde{x}_i$, or $\bar{x}_i$, by the substitutions

$$\tilde{x}_i = \epsilon x_i^* \qquad \text{(Outer variables)} \qquad (2\text{-}6a)$$

$$\tilde{x} = \bar{x},\ \tilde{y} = \epsilon^{1/2}\bar{y},\ \tilde{z} = \epsilon^{1/2}\bar{z} \qquad \text{(Inner variables)} \qquad (2\text{-}6b)$$

The parameter ϵ and the variables $\tilde{x}_i$ admit the following evident interpretation: $\tilde{x}_i = x_i/R$ are the coordinates of a point referred to an extraneous standard of length measurement, of length R. In the outer limit process, R and $\tilde{x}_i$ are fixed while the characteristic length, $L = \epsilon R$ is decreased to zero; the Reynolds number, $Re = UL/\nu$ is held fixed in the process. By repeated applications of the outer and inner limit processes to a given flow quantity W one obtains (provided the limits exist) two expansions, outer and inner, of the form

$$W \sim \sum_{i=0} \delta_i(\epsilon)\, \tilde{w}_i(\tilde{x}_i) \qquad \text{(Outer expansion)} \qquad (2\text{-}7a)$$

$$W \sim \sum_{i=0} \delta_i(\epsilon)\, \bar{w}_i(\bar{x}_i) \qquad \text{(Inner expansion)} \qquad (2\text{-}7b)$$

Here $\{\delta_i(\epsilon)\}$ $(i = 0, 1, 2, \ldots)$ is a sequence of functions (called orders or gauge functions) such that

$$\lim_{\epsilon \to 0} \frac{\delta_{i+1}}{\delta_i} = 0, \quad i = 0, 1, 2, \ldots \qquad (2.\ 7c)$$

If domains of validity of partial sums of expansions 2-7 overlap, as discussed in reference 4, it is then possible to construct a composite expansion which is uniformly valid for $r^* \to \infty$.

The terms of expansions 2-7 are defined by the form of the expansions 2-7a and 2-7b, except for the trivial freedom allowed in the choice of δ's. The "form" of the expansion, understood in an extended sense to include the stipulated domains of uniform validity, also determines the equations, the boundary and matching conditions which the terms must satisfy. (The equations may be found by a formal substitution of the series in equations 2-3.)

In the present thesis, matching series of the form 2-7 will be constructed on the basis of equations, boundary, and matching conditions (and an additional condition, namely that of eliminability of the extraneous parameter, see Section 2. 3). The existence of an actual asymptotic expansion of the form 2-7 is not absolutely essential and is in fact not stipulated in the present thesis; this question is discussed explicitly below. Different quantities, W will be introduced as needed in each case (see equations 3-5, 3-6, 4-1, and 4-4, which give the explicit forms for the several cases treated). The gauge functions, $\delta_i(\epsilon)$, will be determined iteratively, but not in strictly consecutive order; the iteration process involves "switchback" (as does that of Chang (2)). The δ's will be reindexed in the form

$$\delta_i = \delta_{\nu_i} \qquad (2\text{-}7d)$$

where the ν's are chosen as convenient in the iteration process (each ν represents what is regarded as a definite step in the procedure).

The stipulated domains of uniform validity of 2-7a and 2-7b may be described as follows: under the outer limit process $\vec{q}^*$ tends to $\vec{i}$ and p^* tends to zero uniformly over the entire $\tilde{x}$-space excluding the point at the origin $\tilde{x}_i = 0$. This is evident by hypothesis

(i. e. from the boundary conditions 2-5a). However, in general, the outer expansion is not uniform at the positive $\tilde{x}$-axis. This is due to the presence of singular perturbations which represent the decay of the wake and are, in general, of order $\epsilon^{1/2}$ (e. g. in the presence of lift) or of order ϵ (drag but no lift). The inner expansion, on the other hand, should be valid in the wake region, or, more precisely in the right half of the $\bar{x}_i$-space, excluding the plane $\bar{x} = 0$. The regions of validity of 2-7a and 2-7b should overlap for large $\bar{\rho}$ (small $\tilde{\rho}$).* The non-uniformity of the inner expansion at the plane $\bar{x} = 0$ is not important: it is stipulated that the outer expansion is valid at that plane, excluding the point at the origin $\bar{x}_i = 0$. Hence, the two expansions, inner and outer, should match also for small $\bar{x} > 0$. An additional stipulation will be added in order to derive the boundary conditions at infinity for the outer expansion: it is stipulated that the outer expansion is uniform at infinity excluding only the positive $\tilde{x}$-axis, and that the two expansions, outer and inner, jointly cover the point at infinity.

Intended Validity

In this thesis, matching series of the form 2-7 are constructed on the basis of the associated equations and conditions. The results are intended to be valid in the following sense:

(i) For every partial sum of the expansion, and for every choice of the arbitrary constants of the series, there should exist

*See equations 3-2 and 3-4.

a related Navier-Stokes solution of the class defined in Section 2.1 (i.e. an "exact solution").

(ii) Whenever an exact solution has an expansion of the form 2-7, then the expansion should be given correctly by our results.

Statements (i) and (ii) tell us in what sense our series is "correct" or "grossly incorrect." No other questions enter the construction process. The guidance supplied by statements (i) and (ii) is, however, needed in the construction procedure.

In the first place, statement (ii) determines the associated equations and boundary conditions for each term of the series. This has already been discussed in a proceding paragraph (see Section 2.2). In accordance with (ii), therefore, we should admit, for each term of (2-7), the most general expression which is allowed by the boundary, matching, and eliminability conditions (provided only that the expression is not in contradiction with (i)). In the present expansions, there exist complementary solutions of the associated equations (called "eigensolutions") which satisfy homogeneous boundary and matching conditions and the condition of eliminability (see Section 2.3). (Here, by "boundary" and "matching" conditions we understand, of course, those conditions which are <u>governing</u> for the term in question, i.e. those conditions which may be derived from the overlap principle or from condition 2-5a. Thus e.g. it is not required that an outer "eigensolution" vanish at the positive $\tilde{x}$-axis, since no such fact derives from the basic premises.) The most general expression for each term is obtained by finding all possible eigensolutions which are admitted by the governing conditions for the term in question. In particular, it appears that the question of so-called "intermediate orders" or "phantom terms" (i. e. the question of existence of terms of order not listed explicitly in each case) may, in principle, be decided entirely on the basis of the conditions. This is illustrated in Section 3. 4.

In accordance with (i) it is necessary to take into account the possible "integrated effects" of the forcing term: the domains of validity do not become evident until such effects are considered. In general, the "integrated effects" include the possibility of resonance and also other possibilities, e. g. that the solution may be rendered multivalued. In the present case, however, an estimate of the effects of the forcing term is provided at each step by a subsequent term of the series, and will be here, in principle, taken into account by inspection. The forcing terms appear to be entirely harmless within the stipulated domains of validity.

It is believed that no other essential considerations need to be taken into account: although the theory of constructions such as the present ones has never been fully discussed, it has been suggested to the author that a favorable estimate of the possible effects of a small, arbitrary forcing term is probably sufficient, i. e. may lead directly to a rigorous proof of (i).

2. 3. The Principle of Eliminability

All governing conditions for our series are derived from the definition of the exact solutions, by means of the hypotheses on the validity of the series (cf. Section 2. 2). It should be noted, therefore, that the definition has changed: after the parameter has been introduced, we are dealing with functions $\vec{q}^{*}(\tilde{x}_i;\epsilon)$ and $p^{*}(\tilde{x}_i;\epsilon)$ which satisfy the Navier-Stokes equations 2-3, the conditions α), β), and γ) (Section 2.1, equation 2-5), and the following eliminability conditions:

$$\vec{q}^{*}(x_i^{*}\epsilon;\epsilon) = \lim_{\epsilon \to 0} \vec{q}^{*}(x_i^{*}\epsilon;\epsilon) \tag{2-8a}$$

$$p^{*}(x_i^{*}\epsilon;\epsilon) = \lim_{\epsilon \to 0} p^{*}(x_i^{*}\epsilon;\epsilon) \tag{2-8b}$$

Conditions 2-8 state that ϵ is eliminated by the substitution of $x_i^{*}\epsilon$ for $\tilde{x}_i$. Of a partial sum, S_n, of series 2-7a and 2-7b it is then required that, in its stipulated domain of validity, S_n be expressible in the form

$$S_n = f(x_i^{*}) + R(x_i^{*};\epsilon) \tag{2-9a}$$

where

$$\frac{R}{\delta_n(\epsilon)} \text{ is uniformly small} \tag{2-9b}$$

The condition 2-9 will be referred to as the eliminability principle; it is a governing condition for the series.

2.4. The Principle of Rapid (Transcendental) Decay of Vorticity

It is a certain (although possibly unproved) hydrodynamical fact that, for a finite or semi-infinite solid in a uniform stream, the vorticity decays at an exponential rate with distance outside the wake or the boundary layer, as the case may be. It is a corollary to this fact that

The vorticity must also decay transcendentally in every term of our inner expansions 2-7a as $\bar{x} \to 0$. (2-10)

The term "principle of rapid decay" is in general applied to both the theorem and the corollary, and the corollary is also often understood to stipulate $\bar{\rho} \to \infty$ rather than $\bar{x} \to 0$ (cf. equation 2-10). In this thesis, by "principle of rapid decay" we shall understand the corollary 2-10 rather than the theorem, and $\bar{x} \to 0$ rather than $\bar{\rho} \to \infty$. The reason is the following: the corollary is a direct consequence of the matching conditions at the plane $\bar{x} = 0$, which supply the <u>initial conditions</u> for the partial differential equations involved. (Those solutions which decay algebraically have non-zero vorticity at the plane $\bar{x} = 0$ and, hence, cannot be matched to the outer solutions there.)

The "principle of rapid decay" is not a governing condition, but rather a consequence of the governing matching conditions at $\bar{x} > 0$. In the present construction it is almost equally convenient to use either 2-10 or the proper governing conditions, since the equivalence of the two is evident at every step. However, reference to 2-10 makes it possible to reject algebraic solutions at sight.

III. THE ASYMPTOTIC EXPANSIONS FOR THE AXIALLY-SYMMETRIC CASE

3.1. The Principal Expansions

In the present chapter we shall consider solutions which are symmetric about the x^*-axis. The axially-symmetric problem will be treated as a problem in two dimensions (i.e. in two independent variables). For discussion of the axially-symmetric case as a "special case" of the general three-dimensional problem, see Chapter IV.

The velocity field $\vec{q}^*$ may be expressed in the form

$$\vec{q}^* = u^* \vec{i}_x + v^* \vec{i}_\rho + w^* \vec{i}_\theta \tag{3-1}$$

where $(\vec{i}_x, \vec{i}_\rho, \vec{i}_\theta)$ is a left-handed orthonormal set of vectors corresponding to cylindrical polar coordinates x^*, ρ^*, θ, where

$$\rho^* = (y^{*2} + z^{*2})^{1/2}, \quad \theta = \tan^{-1}\left(\frac{y^*}{z^*}\right) \tag{3-2}$$

The governing equations for the axially-symmetric case are obtained by passage to polar coordinates (see equation A-1 of the Appendix) and putting

$$\frac{\partial u^*}{\partial \theta} = \frac{\partial v^*}{\partial \theta} = \frac{\partial w^*}{\partial \theta} = \frac{\partial p^*}{\partial \theta} = 0 \tag{3-3}$$

in equation A-1. The independent variables are then x^* and ρ^*. The corresponding inner and outer variables are

$$\tilde{x} = \epsilon x^*, \quad \tilde{\rho} = \epsilon \rho^* \qquad \text{(Outer)} \tag{3-4a}$$

$$\bar{x} = \epsilon x^*, \quad \bar{\rho} = \epsilon^{1/2} \rho^* \qquad \text{(Inner)} \tag{3-4b}$$

The limit process expansions to be obtained below are of the form:

Outer: $\vec{q}^* = \vec{i} + \epsilon^2 \vec{q}_2(\tilde{x}, \tilde{\rho}) + \epsilon^3 \log \epsilon\, \vec{q}_{3a}(\tilde{x}, \tilde{\rho}) + \epsilon^3 \vec{q}_3(\tilde{x}, \tilde{\rho})$

$$+ o(\epsilon^3) \qquad (3\text{-}5a)$$

$$p^* = \epsilon^2 \tilde{p}_2(\tilde{x}, \tilde{\rho}) + \epsilon^3 \log \epsilon\, \tilde{p}_{3a}(\tilde{x}, \tilde{\rho}) + \epsilon^3 \tilde{p}_3(\tilde{x}, \tilde{\rho}) + o(\epsilon^3) \qquad (3\text{-}5b)$$

Inner: $u^* = 1 + \epsilon u_1(\bar{x}, \bar{\rho}) + \epsilon^2 \log \epsilon\, u_{2a}(\bar{x}, \bar{\rho}) + \epsilon^2 u_2(\bar{x}, \bar{\rho})$

$$+ o(\epsilon^2) \qquad (3\text{-}6a)$$

$$\bar{v} = \epsilon^{-1/2} v^* = \epsilon v_1(\bar{x}, \bar{\rho}) + \epsilon^2 \log \epsilon\, v_{2a}(\bar{x}, \bar{\rho}) + \ldots \qquad (3\text{-}6b)$$

$$\bar{w} = \epsilon^{-1/2} w^* = \epsilon w_1(\bar{x}, \bar{\rho}) + \ldots \qquad (3\text{-}6c)$$

3. 2. A Remark Concerning the Outer Expansion: Irrotationality of the Outer Flow Field

We shall show, once and for all, that the outer flow field is irrotational to all finite orders ϵ^n as $\epsilon \to 0$, or, equivalently, to all finite orders r^{*-n} as $r^* \to \infty$.

The general Navier-Stokes equations in outer variables are

$$\vec{q}^* \cdot \tilde{\nabla} \vec{q}^* + \tilde{\nabla} p^* = \frac{\epsilon}{Re} \tilde{\nabla}^2 \vec{q}^* \qquad (3\text{-}7a)$$

$$\tilde{\nabla} \cdot \vec{q}^* = 0 \qquad (3\text{-}7b)$$

$$\tilde{\nabla} = \left(\frac{\partial}{\partial \tilde{x}_i} \right) \qquad (3\text{-}7c)$$

If the outer expansion 3-5 is inserted into 3-7, a term by term calculation may in principle be carried out. It is possible, however, to obtain the result by a direct argument. One first observes that an irrotational, solenoidal vector field $\vec{q}^*$ is a solution of the Navier-Stokes equations. In particular, the term of order ϵ in equation 3-7a is zero for any such solution. I. e. now impose the condition that the vorticity of the related Navier-Stokes solutions be zero at upstream infinity. This determines the outer limit to be an irrotational flow. In the succeeding calculations, the right-hand side of equation 3-7a will always vanish. We conclude that the outer expansion of the vector velocity consists of a series of terms each of which is the gradient of a harmonic function of $\tilde{x}_i$. The outer expansion of pressure is then a consequence of the constancy of total head in potential flows.

In particular, if the expansions 3-5a and 3-5b are inserted into Bernoulli's equation, one finds

$$\tilde{p}_2 = -\vec{q}_2 \cdot \vec{i}, \quad \tilde{p}_{3a} = -\vec{q}_{3a} \cdot \vec{i}, \quad \tilde{p} = -\vec{q}_3 \cdot \vec{i} \qquad (3\text{-}8)$$

3. 3. Equations for the Inner Terms

The exact equations for the axially-symmetric case, written in inner variables, are

$$H(u') + \frac{\partial p^*}{\partial \bar{x}} = -u' \frac{\partial u'}{\partial \bar{x}} - \bar{v} \frac{\partial u'}{\partial \bar{\rho}} + \frac{\epsilon}{Re} \frac{\partial^2 u'}{\partial \bar{x}^2} \qquad (3\text{-}9a)$$

$$\frac{\partial \bar{\rho} u'}{\partial \bar{x}} + \frac{\partial \bar{\rho}\, \bar{v}}{\partial \bar{\rho}} = 0 \qquad (3\text{-}9b)$$

$$\frac{\partial p^*}{\partial \bar{\rho}} = -\epsilon \left[H_1(\bar{v}) + u' \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{\rho}} - \frac{\bar{w}^2}{\bar{\rho}} \right] + \frac{\epsilon^2}{Re} \frac{\partial^2 \bar{v}}{\partial \bar{x}^2} \qquad (3\text{-}9c)$$

$$H_1(\bar{w}) = -u' \frac{\partial \bar{w}}{\partial \bar{x}} - \bar{v} \frac{\partial \bar{w}}{\partial \bar{\rho}} - \frac{\bar{v}\, \bar{w}}{\bar{\rho}} + \frac{\epsilon}{Re} \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \qquad (3\text{-}9d)$$

where u', $\overline{v}$, and $\overline{w}$ are defined by

$$u' = u^* - 1, \quad \overline{v} = \epsilon^{-1/2} v^*, \quad \overline{w} = \epsilon^{-1/2} w^* \tag{3-9e}$$

and the linear differential operators H and H_1 are defined by

$$H = \frac{\partial}{\partial \overline{x}} - \frac{1}{Re}\left(\frac{\partial^2}{\partial \overline{\rho}^2} + \frac{1}{\overline{\rho}}\frac{\partial}{\partial \overline{\rho}}\right) \tag{3-9f}$$

$$H_1 = H + \frac{1}{Re}\frac{1}{\overline{\rho}^2} \tag{3-9g}$$

Each term of 3-9a-d tends to a uniform limit under the inner limit process. By a repeated application of the inner limit process to the exact governing equations, or equivalently, by formal substitution of the inner expansion, one obtains governing "approximate" equations satisfied by the terms of the inner expansion. The approximate partial differential equations are

$$H(u_\nu) + \frac{\partial p_\nu}{\partial \overline{x}} = f_\nu \tag{3-10a}$$

$$\frac{\partial \overline{\rho}\, u_\nu}{\partial \overline{x}} + \frac{\partial \overline{\rho}\, \overline{v}_\nu}{\partial \overline{\rho}} = 0 \tag{3-10b}$$

$$\frac{\partial p_\nu}{\partial \overline{\rho}} = g_\nu \tag{3-10c}$$

$$H_1(w_\nu) = h_\nu \tag{3-10d}$$

Note that ν is not necessary an integer (cf. 3-6). The forcing terms f_ν, g_ν, h_ν vanish for $\nu < 2$. Equations 3-10a-c form a simultaneous system; 3-10d may be solved independently.

3. 4. The Leading Terms of the Inner and Outer Expansions: u_1, v_1, w_1; p_1; $\vec{q}_2$

The equations for u_1, v_1, w_1 and p_1 are

$$H(u_1) + \frac{\partial p_1}{\partial \overline{x}} = 0 \tag{3-11a}$$

$$\frac{\partial \overline{\rho}\, u_1}{\partial \overline{x}} + \frac{\partial \overline{\rho}\, v_1}{\partial \overline{\rho}} = 0 \tag{3-11b}$$

$$\frac{\partial p_1}{\partial \overline{\rho}} = 0 \tag{3-11c}$$

$$H_1(w_1) = 0 \tag{3-11d}$$

The relevant solutions are

$$u_1 = -\frac{a\,Re}{4\pi}\,\frac{e^{-\sigma}}{\overline{x}}, \quad v_1 = -\frac{a}{2\pi}\,\frac{\sigma}{\overline{\rho}\,\overline{x}}\,e^{-\sigma} \tag{3-12a}$$

$$w_1 = -\frac{m\,Re}{4\pi}\,\frac{\sigma}{\overline{\rho}\,\overline{x}}\,e^{-\sigma}, \quad p_1 = 0 \tag{3-12b}$$

where

$$\sigma = \frac{Re}{4}\,\frac{\overline{\rho}^2}{\overline{x}} = \frac{Re}{4}\,\frac{\rho^{*2}}{x^*} \tag{3-12c}$$

and a and m are arbitrary constants. As a consequence of condition 2-5b, the constant a may be related to the dimensionless drag experienced by any closed streamsurface; the constant m may be related to the moment (see Section A. 4). The constant a also represents the strength of a "viscous sink" placed at the origin. This may be seen by considering the streamfunction $\psi_1(\overline{x},\overline{\rho})$ for the terms u_1 and v_1:

$$\psi_1 = \frac{a}{2\pi} e^{-\sigma} \; ; \quad u_1 = \frac{1}{\bar{\rho}} \frac{\partial \psi_1}{\partial \bar{\rho}} \; , \quad v_1 = - \frac{1}{\bar{\rho}} \frac{\partial \psi_1}{\partial \bar{x}} \tag{3-13a}$$

One sees that, for $\bar{x} > 0$,

$$2\pi[\psi_1(\bar{x}, \infty) - \psi_1(\bar{x}, 0)] = -a \tag{3-13b}$$

To obtain 3-12, one first observes that the principle of eliminability requires that p_1 be a constant multiple of $\bar{x}^{-1}$. By matching with the outer expansion 3-5b, it follows that $p_1 = 0$. Also, again by eliminability,

$$u_1 = \frac{1}{\bar{x}} f(\sigma) \tag{3-14}$$

We require that $f(\sigma)$ be regular at $\sigma = 0$ and vanish exponentially as $\sigma \to \infty$. Inserting 3-14 into 3-11a, one obtains a second order ordinary differential equation for f. It is shown in Section A. 2 that f is determined by the conditions stated above to be a constant multiple of $e^{-\sigma}$. This determines u_1 to within a multiplicative constant. v_1 may then be found by integrating the continuity equation; the constant of integration is zero since v_1 is regular on the line $\bar{\rho} = 0$. The calculation of w_1 is similar to that of u_1 and will be omitted.

The outer terms of order ϵ^2 appear as a consequence of the existence of a non-zero drag. $\vec{q}_2$ is determined by requiring that the mass flux through any closed surface containing the solid be zero. This condition has been stated above by equation 2-5b. In order to balance the mass inflow in the wake (cf. equation 3-13b) a term representing the flow due to a potential source must appear in the outer expansion of $\vec{q}^*$. One finds

$$\vec{q}_2 = -\frac{a}{4\pi}\tilde{\nabla}\left(\frac{1}{\tilde{r}}\right), \quad \tilde{r} = (\tilde{x} + \tilde{\rho}^2)^{1/2} \qquad (3\text{-}15)$$

One notes that, if condition 2-5b were relaxed, the multiplicative constant in 3-15 would be arbitrary.

The possibility of terms of orders other than those exhibited explicitly in 3-5 and 3-6 will not be discussed in full detail. However we shall eliminate terms of all orders ϵ^a $(0 \leq a < 2, \quad a \neq 1)$. This is accomplished by referring to the principle of transcendental decay: First we shall consider the outer expansion. Our solutions must be regular except at 0 and ∞ and possibly along the positive $\tilde{x}$-axis. It is evident also that the outer solutions are regular on the positive $\tilde{x}$-axis (except possibly at ∞) unless they match with non-trivial solutions of the homogeneous system 3-11. Hence it is sufficient to restrict attention to (i) the homogeneous system 3-11 and (ii) solutions of Laplace's equation which are regular everywhere except at the origin and ∞. We next discover that the outer solutions are regular at ∞ since there the condition $\vec{q}^* = \vec{i}$ must be satisfied (in virtue of a basic stipulated regions of validity of the outer expansion). Hence the solutions of Laplace's equation are poles and proceed in integral powers of $\tilde{r}^{-1}$ or equivalently in integral powers of ϵ.

Turning next to the homogeneous system 3-11, we shall denote inner terms of order ϵ^a by subscript a. One sees immediately that p_a is a multiple of $\bar{x}^{-a}$. By matching with the outer expansion, we conclude $p_a = 0$ $(0 \leq a < 2)$. Also, we have $u_a = \bar{x}^{-a} f_a(\sigma)$. From the results of Section A. 2 one finds that if $p_a = 0$ and u_a satisfies the condition of transcendental decay, then necessarily $a = 1$. An analogous statement holds for w_a.

3. 5. The Inner Terms of Order ϵ^2: u_2, v_2, w_2, p_2.

In this section the terms u_2, v_2, w_2, and p_2 will be given. The relevant equations are

$$H(u_2) + \frac{\partial p_2}{\partial \bar{x}} = \frac{1}{Re} \frac{\partial^2 u_1}{\partial \bar{x}^2} + f_2 \tag{3-16a}$$

$$\frac{\partial \bar{\rho} u_2}{\partial \bar{x}} + \frac{\partial \bar{\rho}\, v_2}{\partial \bar{\rho}} = 0 \tag{3-16b}$$

$$\frac{\partial p_2}{\partial \bar{\rho}} = 0 \tag{3-16c}$$

$$H_1(w_2) = \frac{1}{Re} \frac{\partial^2 w_1}{\partial \bar{x}^2} + h_2 \tag{3-16d}$$

where the forcing functions f_2 and h_2 are defined by

$$f_2 = -\left(u_1 \frac{\partial u_1}{\partial \bar{x}} + v_1 \frac{\partial u_1}{\partial \bar{\rho}}\right) = a^2 \frac{e^{-2\sigma}}{\bar{x}^3} \tag{3-17a}$$

$$h_2 = -\left(u_1 \frac{\partial w_1}{\partial \bar{x}} + v_1 \frac{\partial w_1}{\partial \bar{\rho}} + \frac{v_1 w_1}{\bar{\rho}}\right) = a\beta \frac{\sigma e^{-2\sigma}}{\bar{\rho}\, \bar{x}^3} \tag{3-17b}$$

The constants appearing in the last equations are not new and are given by

$$a = \frac{a\, Re}{4\pi} \quad , \quad \beta = \frac{m\, Re}{4\pi} \tag{3-17c}$$

The correct solution of 3-16c is

$$p_2 = c\bar{x}^{-2} \tag{3-18}$$

where c is a constant. Noting that the inner expansion of $\vec{i}\cdot\vec{q}_2$ is

$$\vec{i}\cdot\vec{q}_2 = \frac{a}{4\pi}\frac{1}{\bar{x}^2} + O(\epsilon) \tag{3-19}$$

one sees that, by matching,

$$c = -\frac{a}{4\pi}$$

Solving now for u_2, it is convenient to divide the forcing term f_2 into two parts:

$$f_2^{(1)} = \frac{a^2}{\bar{x}^3}\left[e^{-2\sigma} + \frac{1}{4}(\sigma-1)e^{-\sigma}\right] \tag{3-21a}$$

$$f_2^{(2)} = \frac{a^2}{4\bar{x}^3}(1-\sigma)e^{-\sigma} \tag{3-21b}$$

If $f_2^{(1)}$ appears in place of f_2 in 3-16a, there exists a particular solution from which the parameter is strictly eliminable. It is

$$u_2^{(1)} = \frac{a}{4\pi}\frac{1}{\bar{x}^2} + \frac{\bar{x}}{Re}\frac{\partial^2 u_1}{\partial\bar{x}^2} + \frac{a^2}{4}\frac{1}{\bar{x}^2}\left\{(1-\sigma)e^{-\sigma}\left[Ei(-\sigma) - \log\sigma\right] - 2e^{-\sigma} - e^{-2\sigma}\right\} \tag{3-22}$$

The notation for the exponential integral is that of reference 5. A particular integral for the forcing term $f_2^{(2)}$ is

$$\frac{a^2}{4}\frac{\log\bar{x}}{\bar{x}^2}(1-\sigma)e^{-\sigma} \tag{3-23}$$

as is easily seen from the fact that

$$\frac{1}{\overline{x}^2}(1-\sigma)e^{-\sigma} \tag{3-24}$$

is an eigensolution. Using this fact, we arrive at the general solution of 3-16a:

$$u_2 = u_2^{(1)} + u_2^{(2)} \tag{3-25a}$$

$$u_2^{(2)} = \frac{a^2}{4}\,\frac{\log \overline{x}}{\overline{x}^2}(1-\sigma)e^{-\sigma} + \frac{a_1}{\overline{x}^2}(1-\sigma)e^{-\sigma} \tag{3-25b}$$

where $u_2^{(1)}$ is given by 3-22 and a_1 is an arbitrary constant. One sees that, due to the presence of the term involving $\log \overline{x}$ in 3-25b, ϵ is not eliminable from $\epsilon^2 u_2$. This will necessitate the introduction of a term of intermediate order (see Section 3.6).

Integrating the continuity equation and adjusting the constant of integration so as to make v_2 regular at $\overline{\rho} = 0$, one finds

$$v_2 = v_2^{(1)} + v_2^{(2)} \tag{3-26a}$$

where

$$v_2^{(1)} = \frac{a}{4\pi}\frac{\overline{\rho}}{\overline{x}^3} + \frac{\overline{x}}{Re}\,\frac{\partial^2 v_1}{\partial \overline{x}^2} + \frac{a^2}{2\,Re}\,\frac{\sigma}{\overline{\rho}\,\overline{x}^2}\{(2-\sigma)e^{-\sigma}[\,Ei(-\sigma) - \log\sigma]$$

$$+ \frac{1}{\sigma}(e^{-\sigma} + e^{-2\sigma} - 2) - 3e^{-\sigma} - e^{-2\sigma}\} \tag{3-26b}$$

$$v_2^{(2)} = \frac{2a_1}{Re}\,\frac{\sigma}{\overline{\rho}\,\overline{x}^2}(2-\sigma)e^{-\sigma} + \frac{a^2}{2\,Re}\,\frac{\log \overline{x}}{\overline{\rho}\,\overline{x}^2}\,\sigma(2-\sigma)e^{-\sigma} \tag{3-26c}$$

In order to determine w_2, one first notes that if we define

$$u_p = \frac{\beta}{a} u_2^{(1)} + \frac{a\beta}{4} \frac{\log \bar{x}}{\bar{x}^2} (1-\sigma)e^{-\sigma} \tag{3-27a}$$

then

$$\frac{\partial}{\partial \bar{\rho}} H(u_p) = H_1\left(\frac{\partial}{\partial \bar{\rho}} u_p \right) = \frac{\beta}{a} \frac{\partial f_2}{\partial \bar{\rho}} = -4h_2 \tag{3-27b}$$

Thus

$$w_2 = -\frac{1}{4} \frac{\partial u_p}{\partial \bar{\rho}} + \frac{\bar{x}}{Re} \frac{\partial^2 w_1}{\partial \bar{x}^2} + m_1 \frac{\sigma}{\bar{\rho}\,\bar{x}^2} (2-\sigma)e^{-\sigma} \tag{3-28}$$

where m_1 is an arbitrary constant. Note that w_2 decays exponentially as $\bar{x} \to 0$. It is evident also that every partial sum of the inner expansion of $\bar{w}$ decays exponentially, since otherwise the pressure would be multivalued.

3.6. Switchback. Terms of Intermediate Order: u_{2a}, v_{2a}, w_{2a}.

We have noted above that the parameter is not strictly eliminable from the terms $\epsilon^2 u_2$, $\epsilon^{5/2} v_2$, $\epsilon^{5/2} w_2$. For example,

$$\epsilon^2 u_2(\bar{x}, \bar{\rho}) = u_2(x^*, \rho^*) + \log \epsilon \cdot \frac{a^2}{4} \frac{1}{x^{*2}} (1-\sigma)e^{-\sigma} \tag{3-29}$$

On the other hand, ϵ is eliminable from

$$\epsilon^2 u_2(\bar{x}, \bar{\rho}) - \epsilon^2 \log \epsilon \cdot \frac{a^2}{4} \frac{1}{\bar{x}^2} (1-\sigma)e^{-\sigma} \tag{3-30}$$

The second term appearing in 3-30, of order $\epsilon^2 \log \epsilon$, is precisely the term u_{2a} occurring in 3-6a. Terms which arise in this manner will be referred to as "switchback terms." Other examples of switchback have been discussed by Chang (2). In our construction, switchback terms are uniquely determined by the principle of eliminability.

No attempt will be made to explain precisely the reason for switchback. It should be noted, however, that it is here a non-linear phenomenon. Switchback terms do not appear in the expansions of Oseen solutions. Also, one observes that a particular integral for a forcing term which is an eigensolution will always require a switchback term (cf. equation 3-21b, see also reference 2, Appendix).

By writing relations analogous to 3-29 for v_2 and w_2, the switchback terms v_{2a} and w_{2a} may be found. In each case the switchback term is observed to be a solution of the homogeneous equation. This can be checked by deriving the equations for the inner terms of order $\epsilon^2 \log \epsilon$ in the usual way.

3. 7. The Outer Term of Order ϵ^3: $\vec{q}_3$.

One finds from 3-26 that the outer expansion of the inner expansion of v^* is

$$\tilde{v} = \epsilon^2 \frac{a}{4\pi} \frac{\tilde{\rho}}{\tilde{x}^3} - \epsilon^3 \frac{a^2}{Re} \frac{1}{\tilde{\rho}\,\tilde{x}^2} + o(\epsilon^3) \tag{3-31}$$

The first term represents the flow in the wake due to the potential source and matches with $\vec{q}_2$; the second matches with a term of order ϵ^3 in the outer expansion of $\vec{q}^*$. This term is

$$\vec{q}_3 = \tilde{\nabla} \phi_3 , \quad \tilde{\nabla}^2 \phi_3 = 0 \tag{3-32a}$$

where

$$\phi_3 = \frac{a^2}{2Re}\frac{1}{\tilde{r}^2}[\tilde{x}\log(\tilde{r}-\tilde{x}) + \tilde{r} - 2\tilde{x}\log\tilde{r}] + \frac{\tilde{a}_1}{2}\frac{\tilde{x}}{\tilde{r}^3} \qquad (3\text{-}32b)$$

The first term on the right of 3-32b is required by matching, and is associated with the switchback term $\vec{q}_{3a}$. The second is a potential dipole of arbitrary strength; this dipole is the eigensolution of order ϵ^3, i.e. it is the most general harmonic function homogeneous of degree 2 in the variables $\tilde{x}_i$ and regular everywhere except at the origin.

3.8. The Inner Term p_3.

The term of order ϵ^3 in the inner expansion of pressure satisfies

$$\frac{1}{\bar{\rho}}\frac{\partial}{\partial\bar{\rho}}(\bar{\rho}\frac{\partial p_3}{\partial\bar{\rho}}) = -\frac{\partial^2 p_2}{\partial\bar{x}^2} + \frac{\partial f_2}{\partial\bar{x}} + \frac{1}{\bar{\rho}}\frac{\partial}{\partial\bar{\rho}}(\bar{\rho}\, f_2') \qquad (3\text{-}33)$$

where f_2 is given by 3-17a and

$$f_2' = -u_1\frac{\partial v_1}{\partial\bar{x}} - v_1\frac{\partial v_1}{\partial\bar{\rho}} + \frac{w_1^2}{\bar{\rho}} \qquad (3\text{-}34a)$$

$$= (\frac{3a^2}{Re} + \frac{Re\,\beta^2}{4})\frac{\sigma}{\bar{\rho}\,\bar{x}^3}e^{-2\sigma} \qquad (3\text{-}34b)$$

One may obtain 3-33 by taking the divergence of 2-3a, expressing the resulting equation in inner variables, and expanding. The solution is

$$p_3 = \frac{3a}{8\pi}\frac{\bar{\rho}^2}{\bar{x}^4} + \frac{a^2}{2Re}[2Ei(-2\sigma) - \frac{1}{2}e^{-2\sigma} - 2\log\sigma]$$

$$-\frac{Re}{16}\beta^2\frac{e^{-2\sigma}}{\bar{x}^3} + c_1\frac{\log\bar{x}}{\bar{x}^3} + \frac{c_2}{\bar{x}^3} \qquad (3\text{-}35)$$

The constants c_1 and c_2 can be determined by matching. One finds by expanding in the overlap domain that the inner and outer expansions of p^* match to order ϵ^3 inclusive if and only if

$$c_1 = \frac{2\alpha^2}{\mathrm{Re}} \tag{3-36a}$$

$$c_2 = \frac{\alpha^2}{2\,\mathrm{Re}}(\log \mathrm{Re} - 9) + \tilde{a}_1 \tag{3-36b}$$

where $\tilde{a}_1$ is the arbitrary constant appearing in 3-32b. Note that 3-35 is associated with a switchback term p_{3a}.

The term p_3 leads to the following conclusions regarding the inner expansion of p^* for the axially-symmetric case: (i) to order ϵ^3 exclusive, the pressure penetrates the wake, i. e. all terms are functions of $\bar{x}$ alone, (ii) due to the non-linear effect, there exists a term of order ϵ^3 which is discontinuous across the wake; the pressure discontinuity of this order is

$$p_3'(\bar{x}, \infty) - p_3'(\bar{x}, 0) = \frac{1}{\bar{x}^3}\left[\frac{\mathrm{Re}\,\beta^2}{16} - \frac{\alpha^2}{2\,\mathrm{Re}}(4\gamma + 4\log 2 - 1)\right] \tag{3-37a}$$

where

$$p_3' = p_3 - \frac{3a}{8\pi}\frac{\bar{\rho}^2}{\bar{x}^4} \tag{3-37b}$$

and γ = Euler's constant.

3.9. Higher Order Terms.

Terms of higher order have not been studied in detail. However, it appears that the construction proceeds indefinitely, involving no change of the basic form (3-5, 3-6) of the expansions. A suitable sequence of orders, $\{\delta_k(\epsilon)\}$, seems to consist of functions $\epsilon^i(\log \epsilon)^j$, where j is bounded above by $i-1$. It has been brought to our attention that a similar conjecture seemed justified in reference 2.

Partial sums of the inner and outer expansions involve a number of arbitrary constants (e. g. a, m, a_1, $\tilde{a}_1$, m_1). A general fact may be pointed out concerning these constants: the number of arbitrary constants and the eigensolutions are unchanged if the Navier-Stokes equations are replaced by the Oseen equations. Hence, insofar as our construction procedure indicates, given any Navier-Stokes solution which has an expansion of the form considered here, one can find an Oseen solution with the same constants, and conversely.

IV. THE ASYMPTOTIC EXPANSIONS FOR THE GENERAL THREE-DIMENSIONAL CASE

4.1. Cross Flow and Pressure: The Multidimensional Problem

In the present chapter, expansion procedures are given and expansions are constructed for the general class of three-dimensional Navier-Stokes solutions defined in Chapter I (see Section 2.1). Axial symmetry is no longer required. However, a different expansion procedure suggests itself (and will be here adopted), due to certain differences in the nature of cross flow and pressure in the wake region.

For problems involving a two-dimensional continuity equation (e.g. (2) or Chapter III), whenever u^* is given, the order of the cross flow (and, indeed, the cross flow itself) is fully determined by the continuity equation alone, together with a suitable boundary or matching condition. Consider, e.g. the axially symmetric case: here $u^* - 1$ is of order ϵ in the wake region. $\vec{q}^* - 1$ is of order ϵ^2 in the outer region. Hence, by the continuity equation and matching (!), v^* is of order $\epsilon^{3/2}$ in the wake. On the other hand, v^* is "small" with respect to the momentum equation, which then degenerates to $\partial p^*/\partial \overline{\rho} = 0$.

For a problem involving a multidimensional continuity equation, however, the situation is different. Here, the continuity equation does not determine the cross flow. Hence, the cross flow may be large with respect to the continuity equation and also with respect to the matching conditions for large $\overline{\rho}$. A striking illustration is given by the so-called "lifting case" (see Section 4.4). For the lifting case, the axial velocity disturbance, $u^* - 1$, remains of order ϵ (equivalently, r^{*-1}) in the wake region. The cross-flow components, v^* and w^*, are, however, also of order ϵ and, hence, are obviously "large" with respect to the continuity equation. The large cross flow in the lifting case arises from the "horseshoe vortex" term of the wake expansion, $\vec{\bar{q}}_1^{\,+}$ (equation 4-15b), which, in particular, vanishes for large $\bar{\rho}$ and therefore satisfies homogeneous matching conditions. On the other hand, v^* and w^* are no longer "small" with respect to the momentum equation: the governing system of approximate equations is a system of three simultaneous partial differential equations for v^*, w^*, and p^*, namely the two momentum equations (4-6b) and the continuity equation (4-6c) in the cross-flow plane. Hence, procedures for the multi-dimensional case involve a different sequence of steps, which, in a sense, is opposite to that of Chapter III and reference 2.

The three-dimensional procedures are introduced in Sections 4.2 and 4.3, below, by a statement of form of the expansion (equations 4-1 and 4-4), and explained further in subsequent sections. The methods of Chapter III are certainly not a "special case" of the multi-dimensional methods: solutions with axial symmetry may, of course, be treated also by the three-dimensional method and the results must agree numerically. However, the construction is entirely different. This, however, engenders no paradox, since in Chapter III axial symmetry is introduced into the governing equations, while in the present chapter it is a consequence of the boundary conditions.

4. 2. The Inner and Outer Expansions

The three-dimensional solutions of equations 2-3, Chapter II will be studied. Given the solution of 2-3, the x^*-axis is parallel to the velocity at infinity (see condition 2-5a). The (x^*, y^*) plane will be chosen parallel to the total force which acts upon the body, or at least upon a closed surface in the flow field. The dimensionless variables x_i^*, $\tilde{x}_i$, and $\overline{x}_i$ are defined in Chapter II (see equation 2-6). Under the outer limit process ($\tilde{x}_i$ fixed), $\vec{q}^* \rightarrow \vec{i}$ uniformly. An expansion of the form

$$\vec{q}^* = \vec{i} + \epsilon^2 \vec{q}_2(\tilde{x}_i) + \epsilon^3 \log \epsilon \, \vec{q}_{3a}(\tilde{x}_i) + \epsilon^3 \vec{q}_3(\tilde{x}_i) + o(\epsilon^3) \qquad (4\text{-}1a)$$

$$p^* = \epsilon^2 \tilde{p}_2(\tilde{x}_i) + \epsilon^3 \log \epsilon \, \tilde{p}_{3a}(\tilde{x}_i) + \epsilon^3 \tilde{p}_3(\tilde{x}_i) + o(\epsilon^3) \qquad (4\text{-}1b)$$

shall be constructed. The outer expansion is in general non-uniform in terms of order ϵ or higher at the positive $\tilde{x}$-axis. The non-uniformity represents the wake. The inner expansion will be treated differently than in the axially-symmetric case. The velocity $\vec{q}^*$ shall be expressed as the sum of a cross-flow velocity $\vec{q}^+$ and an axial velocity $u\vec{i}$:

$$\vec{q}^* = u\vec{i} + \vec{q}^+ \qquad (4\text{-}2)$$

In order to obtain an indexing of terms such that each value of the index corresponds to a definite "step" in the construction, we introduce new dependent variables

$$\overline{u} = \epsilon^{-1/2}(u^* - 1), \quad \overline{p} = \epsilon^{-1/2} p^* \tag{4-3}$$

For $\overline{x}_i$ fixed, an expansion of the form

$$\overline{u} = \epsilon^{1/2} u_{1/2}(\overline{x}_i) + \epsilon \log \epsilon \, u_{1a}(\overline{x}_i) + \epsilon u_1(\overline{x}_i) + o(\epsilon) \tag{4-4a}$$

$$\vec{q}^{\,+} = \epsilon \vec{q}_1^{\,+}(\overline{x}_i) + \epsilon^{3/2} \log \epsilon \, \vec{q}^{\,+}_{3/2a}(\overline{x}_i) + \epsilon^{3/2} \vec{q}_{3/2}(\overline{x}_i) + o(\epsilon^{3/2}) \tag{4-4b}$$

$$\overline{p} = \epsilon^{3/2} p_{3/2}(\overline{x}_i) + o(\epsilon^{3/2}) \tag{4-4c}$$

will be constructed. In particular the plane of the variables $\overline{y}$ and $\overline{z}$ will be referred to as the "cross-flow plane"; $\overline{x}$ may be regarded as a parameter in the inner expansion.

As explained in Chapter II (see Section 3. 2), the terms of the outer expansion satisfy the Laplace equation,

$$\tilde{\nabla} \times \vec{q}_\nu = \tilde{\nabla} \cdot \vec{q}_\nu = 0 \tag{4-5}$$

The terms of the inner expansion satisfy the following equations:

$$\left(\frac{\partial}{\partial \overline{x}} - \frac{1}{Re} \nabla_+^2\right) u_\nu + \frac{\partial p_\nu}{\partial \overline{x}} = f_\nu \tag{4-6a}$$

$$\left(\frac{\partial}{\partial \overline{x}} - \frac{1}{Re} \nabla_+^2\right) \vec{q}_\nu^{\,+} + \nabla_+ p_\nu = g_\nu \tag{4-6b}$$

$$\nabla_+ \cdot \vec{q}_\nu^{\,+} = h_\nu \tag{4-6c}$$

where ∇_+ denotes the operator

$$\nabla_+ = \left(\frac{\partial}{\partial \overline{y}}, \frac{\partial}{\partial \overline{z}} \right) \tag{4-7}$$

in the cross-flow plane. The forcing terms f_ν, g_ν, and h_ν $(\nu > 0)$, may be determined reiteratively from the exact equations A-1. In particular f_ν vanishes for $\nu < 1$, and g_ν and h_ν vanish for $\nu < 3/2$. Equations 4-6b, c constitute a simultaneous system for the cross-flow and the pressure. At each step the cross-flow and the pressure may be determined first, and then the axial velocity may be found from 4-6a.

4. 3. Fourier Analysis of the Inner Terms. Eigensolutions.

It will be convenient to carry out the solution of 4-6 in cylindrical polar coordinates (see Section 3. 1). In particular, each term of the inner expansion may then be expressed as a terminating trigonometric series in θ, the coefficients of the series depending only upon $\overline{x}$ and $\overline{\rho}$. We define the cross-flow terms v_ν and w_ν by

$$\vec{q}_\nu^{\,+} = v_\nu \vec{i}_\rho + w_\nu \vec{i}_\theta \qquad (4\text{-}8)$$

If $F_\nu(\overline{x}, \overline{\rho}, \theta)$ denotes any of the terms u_ν, v_ν, w_ν, or p_ν expressed as a function of the variables $\overline{x}$, $\overline{\rho}$, and θ, we shall assume that there exists a Fourier expansion of F_ν of the form

$$F_\nu(\overline{x}, \overline{\rho}, \theta) = F_\nu^o(\overline{x}, \overline{\rho}) + \sum_{n=1}^{N} [F_\nu^n(\overline{x}, \overline{\rho})\sin n\theta + \underline{F}_\nu^n(\overline{x}, \overline{\rho})\cos n\theta] \qquad (4\text{-}9)$$

where N is a suitable upper bound, depending upon ν (see Section 4. 5). F_ν^n and $\underline{F}_\nu^n$ will be referred to as the orthogonal Fourier coefficients of order ν and degree n. The Fourier expansion 4-9 will be constructed for several of the inner terms (see Sections 4. 4-4. 8).

The Fourier coefficients u_ν^n, v_ν^n, $\underline{w}_\nu^n$ and p_ν^n $(n = 1, 2, \ldots)$ satisfy the following system of equations:

$$H_n(u_\nu^n) + \frac{\partial p_\nu^n}{\partial \overline{x}} = f_\nu^n \tag{4-10a}$$

$$H_n(\overline{\rho}\, v_\nu^n) + \overline{\rho}\, \frac{\partial p_\nu^n}{\partial \overline{\rho}} = g_\nu^n \tag{4-10b}$$

$$\frac{\partial \overline{\rho}\, v_\nu^n}{\partial \overline{\rho}} \mp n\, \underline{w}_\nu^n = h_\nu^n \tag{4-10c}$$

$$L_n(p_\nu^n) = k_\nu^n \tag{4-10d}$$

where the differential operators H_n and L_n are defined by

$$H_n = \frac{\partial}{\partial \overline{x}} - \frac{1}{Re} L_n \tag{4-10e}$$

$$L_n = \frac{\partial^2}{\partial \overline{\rho}^2} + \frac{1}{\overline{\rho}} \frac{\partial}{\partial \overline{\rho}} - \frac{n^2}{\overline{\rho}^2} \tag{4-10f}$$

The lower sign in 4-10c is understood to apply to the equations for the orthogonal coefficients, obtainable from 4-10 by replacing u_ν^n by $\underline{u}_\nu^n$, $\underline{w}_\nu^n$ by w_ν^n, f_ν^n by $\underline{f}_\nu^n$, etc. The forcing terms f_ν^n, g_ν^n, h_ν^n, $\underline{f}_\nu^n$, etc. $(n = 1, 2, \ldots)$ may be computed from the Fourier expansions of the forcing terms in 4-6.

The case $n = 0$ must be treated somewhat differently, since 4-10 is then a redundant system of equations. Equation 4-10b is in this case replaced by the equation for w_ν^o:

$$H_1(w_\nu^o) = g_\nu^o \tag{4-11}$$

Equations 4-10a, 4-10c, and 4-10d may be solved for u^o_ν, v^o_ν, and p^o_ν, and w^o_ν is given separately by 4-11.

The eigensolutions of the "inner equations" are defined to be the relevant solutions of the homogeneous system of equations corresponding to 4-6, that is, homogeneous solutions satisfying the principles of eliminability and transcendental decay of vorticity (see Chapter II). If the eigensolutions are expanded in Fourier series of the form 4-9, there exists, as a consequence of these principles, a condition on the order and degree of the coefficients: an eigensolution which is of order $\delta_\nu(\epsilon) = \epsilon^\nu$ in the inner expansion 4-4 will have a non-zero Fourier coefficient of degree n $(n = 0, 1, 2, \ldots)$ if and only if

$$\nu - n/2 - 1/2 = \text{non-negative integer} \tag{4-12}$$

This result follows from the discussion in Section A. 2 of the Appendix. Whenever condition 4-12 is satisfied, then the Fourier coefficients of the eigensolutions (denoted below by subscript h) are defined by

$$ {}_h u^n_\nu = a^n_\nu W^n_{\nu+1/2} - {}_h p^n_\nu \tag{4-13a}$$

$$ {}_h v^n_\nu = \frac{b^n_\nu}{\bar\rho} W^n_{\nu-1/2} + \frac{\bar x}{\bar\rho}\, {}_h p^n_\nu \tag{4-13b}$$

$$ {}_h w^n_\nu = \pm \frac{1}{n} \frac{\partial}{\partial \bar\rho} (\bar\rho\, {}_h v^n_\nu) \tag{4-13c}$$

$$ {}_h p^n_\nu = \tilde a^n_\nu\, \bar\rho^{\,n}\, \bar x^{-(\nu+n/2+1/2)} \tag{4-13d}$$

whenever n = positive integer, and by

$$ {}_{h}u^{o}_{\nu} = a^{o}_{\nu} W^{o}_{\nu+1/2} - {}_{h}p^{o}_{\nu} \tag{4-14a}$$

$$ {}_{h}v^{o}_{\nu} = 0 \tag{4-14b}$$

$$ {}_{h}w^{o}_{\nu} = b^{o}_{\nu} W^{1}_{\nu} \tag{4-14c}$$

$$ {}_{h}p^{o}_{\nu} = \tilde{a}^{o}_{\nu}\, \bar{x}^{-(\nu+1/2)} \tag{4-14d}$$

for the case $n = 0$. The eigensolutions W^{n}_{ν} are defined in Section A. 2 (cf. equation A-11). The quantities a^{n}_{ν}, b^{n}_{ν}, and $\tilde{a}^{n}_{\nu}$ $(n = 0, 1, 2, \ldots)$ are arbitrary constants; the orthogonal coefficients involve the constants $\underline{a}^{n}_{\nu}$, $\underline{b}^{n}_{\nu}$, and $\underline{\tilde{a}}^{n}_{\nu}$.

4. 4. The Leading Terms Due to Lift and Drag: $u_{1/2}, \vec{q}_1^{\,+}, \vec{q}_2$.

The leading terms of the inner expansion are

$$ u_{1/2} = u^{o}_{1/2}(\bar{x}, \bar{\rho}) \tag{4-15a}$$

$$ \vec{q}_1^{\,+} = v^{1}_{1}(\bar{x}, \bar{\rho}) \sin \theta + \underline{w}^{1}_{1}(\bar{x}, \bar{\rho}) \cos \theta \tag{4-15b}$$

where

$$ u^{o}_{1/2} = - \frac{a\,Re}{4\pi} \frac{e^{-\sigma}}{\bar{x}} \tag{4-16a}$$

$$ v^{1}_{1} = \frac{b}{2\pi} \frac{1}{\bar{\rho}^{2}} (e^{-\sigma} - 1), \quad w^{1}_{1} = - \frac{b}{2\pi} \frac{1}{\bar{\rho}^{2}} [\,(2\sigma+1)e^{-\sigma} - 1\,] \tag{4-16b}$$

Here a and b are arbitrary constants, related to the drag and lift, respectively (cf. equation 3-12a, see also Section A. 4).

To obtain 4-15 and 4-16, one first notes that the leading terms are eigensolutions, and hence may be constructed from 4-13 and 4-14, subject to condition 4-12. Two cases are significant: (i) $\nu = 1/2$, $n = 0$, and (ii) $\nu = 1$, $n = 1$. The first case gives the leading term in the inner expansion of the axial velocity 4-15a; all other terms are eliminated by matching. The second case gives the leading term of the cross-flow 4-15b; the corresponding pressure term is zero by matching. Note that the coefficients orthogonal to 4-16 vanish as a result of the orientation of the coordinate system (see Section 4. 2).

The term $\vec{q}_2$ of the outer expansion (cf. 3-5a) consists of two parts:

$$\vec{q}_2 = \tilde{\nabla}[\phi_2^o(\bar{x},\bar{\rho}) + \phi_2^1(\bar{x},\bar{\rho})\sin\theta] \tag{4-17a}$$

where

$$\phi_2^o = -\frac{a}{4\pi}\frac{1}{\tilde{r}}, \quad \tilde{r} = (\tilde{x}^2 + \tilde{\rho}^2)^{1/2} \tag{4-17b}$$

$$\phi_2^1 = \frac{b}{4\pi}\frac{\tilde{\rho}}{(\tilde{r}-\tilde{x})\tilde{r}} \tag{4-17c}$$

The potential ϕ_2^o is required by $u^o_{1/2}$, as explained in Section 3. 4, Chapter III. The remaining term in 4-17b is the potential of a "horseshoe vortex" extending downstream from the origin along the positive $\tilde{x}$-axis. The horseshoe vortex term in the outer expansion is required by <u>matching</u> (cf. Section 3. 4), as can be seen from the outer expansion of $\vec{q}_1^{\,+}$.

4.5. A Remark Concerning the Non-linear Effect

We have seen above (Section 4.3) that the order and degree of the coefficients in the Fourier expansion of the eigensolutions of equations 4-6 are subject to the condition

$$\nu - n/a - 1/2 = 0, 1, 2, \ldots \tag{4-18}$$

where n may be any non-negative integer. It can also be shown that the forcing terms in 4-10 vanish unless 4-18 is satisfied. This places an upper limit of $N = 2\nu - 1$ on the degree of the coefficients of order ν in the Fourier expansion of an inner term (cf. equation 4-9). Thus, coefficients of large degree are necessarily of large order.

In order to prove the last assertion, it is convenient to introduce the notation (ν, n) for a term of order ϵ^{ν} in the inner expansion 4-4 whose Fourier expansion involves a non-zero coefficient of degree n $(n = 0, 1, 2, \ldots)$.* The forcing terms in 4-6 may be divided into linear and non-linear parts. As a result of the linear parts, (a, b) generates a higher-order term $(a+1, b)$. The non-linear parts of the forcing terms, however, may generate higher or lower harmonics. In particular, (a, b) and (c, d) may interact non-linearly to generate any of the terms $(a+c+1/2, c+d)$, $(a+c-1/2, c+d)$, $(a+c+1/2, |c-d|)$, or $(a+c-1/2, |c-d|)$.** A simple calculation shows that, whenever

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*The discussion of the present section may easily be extended to include terms of intermediate order, e.g., of order $\epsilon^{\nu}(\log \epsilon)^{\mu}$, $\mu > 0$.

**Note that, as a result of our choice of dependent variables, a term (ν, n) may generate a "higher order" term of the same formal order. The question here is entirely one of numbering the terms in a manner consistent with the possibility of cross-flow. In fact, no such terms appear with respect to the expansions of the cross-flow.

(a, b) and (d, d) satisfy 4-18, so do all terms which they may generate in the higher-order computations. Since the leading terms are eigen-solutions, the assertion follows by induction.

Table 1 shows several terms of integral order which arise as a result of the non-linear effect. The symbol (a, b):(c, d) denotes a term generated by non-linear interaction of (a, b) and (c, d).

ν \ n	0	1	2	3
1/2	Leading terms due to drag			
1		Leading terms due to lift (1/2, 0):(1, 1)		
3/2	(1/2, 0):(3/2, 0) (1, 1):(1, 1)		(1/2, 0):(3/2, 2) (1, 1):(1, 1)	
2		(1/2, 0):(1, 1) (3/2, 0):(1, 1) (3/2, 2):(1, 1) (1/2, 0):(1, 1)		(1/2, 0):(2, 3) (3/2, 0):(1, 1)

It can be shown that, to compute the entry $\nu = a$, $n = b$ in the above table, it is necessary to consider only the terms which appear on or within the upward running diagonals from the point $\nu = a$, $n = b$. Similarly, one finds that the "domain of influence" of a term (a, b) consists of the entries on or within the downward running diagonals.

4.6. The Effect of Wake Displacement: u_1.

The Fourier expansion of the term of order ϵ in the inner expansion of the axial velocity is

$$u_1 = u_1^1(\overline{x}, \overline{\rho}) \sin\theta + \underline{u}_1^1(\overline{x}, \overline{\rho}) \cos\theta \tag{4-19}$$

The coefficients in 4-19 satisfy (cf. equation 4-10a)

$$H_1(u_1^1) + \frac{\partial p_1^1}{\partial \overline{x}} = f_1^1 \tag{4-20a}$$

$$H_1(\underline{u}_1^1) + \frac{\partial \underline{p}_1^1}{\partial \overline{x}} = \underline{f}_1^1 \tag{4-20b}$$

The equation for pressure is homogeneous, and, by matching (cf. equation A-14),

$$p_1^1 = \underline{p}_1^1 = 0 \tag{4-21}$$

The forcing terms in 4-20 are

$$f_1^1 = -\,v_1^1 \frac{\partial u_{1/2}}{\partial \overline{\rho}} = \frac{\alpha\omega}{\overline{\rho}\,\overline{x}^2}\left(e^{-\sigma} - e^{-2\sigma}\right) \tag{4-22a}$$

$$\underline{f}_1^1 = 0 \tag{4-22b}$$

where

$$\alpha = \frac{a\,Re}{4\pi}\,, \qquad \omega = \frac{b\,Re}{4\pi} \tag{4-23}$$

The solution of 4-20 is

$$u_1^1 = \frac{1}{2}\alpha\omega\frac{\sigma}{\overline{\rho}\,\overline{x}}\left[e^{-\sigma}Ei(-\sigma) + \frac{1}{\sigma}\left(e^{-2\sigma} - e^{-\sigma}\right) - e^{-\sigma}\log\sigma + e^{-\sigma}\log\overline{x}\right] + {}_h u_1^1 \tag{4-24a}$$

$$\underline{u}_1^1 = {}_h\underline{u}_1^1 \tag{4-24b}$$

4-24a represents perturbation of the flow due to "displacement of the wake." The non-linear interaction is between the leading terms due to drag and lift. The downwash field associated with the trailing vortex displaces the wake below the positive $\overline{x}$-axis. The first correction for this effect appears in the term u_1. (The cross-flow associated with the deflection of the wake is of higher order, of order ϵ^2.)

The switchback term u_{1a}^1 may be obtained from 4-24a in the usual way (see Section 3.6). The appearance of a term analogous to u_{1a}^1 in the expansion for the two-dimensional case is linked historically with the so-called "Filon Paradox" (see reference 2).

4.7. The Cross-Flow of Order $\epsilon^{3/2}$: $\vec{q}^{\,+}_{3/2}$, $p_{3/2}$

The terms of order $\epsilon^{3/2}$ in the inner expansion of the cross-flow components and pressure have the representations:

$$v_{3/2} = v_{3/2}^o(\overline{x},\overline{\rho}) + v_{3/2}^2(\overline{x},\overline{\rho})\sin 2\theta + \underline{v}_{3/2}^2(\overline{x},\overline{\rho})\cos 2\theta \tag{4-24a}$$

$$w_{3/2} = w_{3/2}^o(\overline{x},\overline{\rho}) + w_{3/2}^2(\overline{x},\overline{\rho})\sin 2\theta + \underline{w}_{3/2}^2(\overline{x},\overline{\rho})\cos 2\theta \tag{4-24b}$$

$$p_{3/2} = p_{3/2}^o(\overline{x},\overline{\rho}) + \underline{p}_{3/2}^2(\overline{x},\overline{\rho})\cos 2\theta \tag{4-24c}$$

The coefficients satisfy the following systems of equations:

$$H_1(w_{3/2}^o) = 0 \tag{4-24a}$$

$$\frac{\partial}{\partial \bar{\rho}} (\bar{\rho}\, v^{o}_{3/2}) = h^{o}_{3/2} \qquad (4\text{-}25b)$$

$$L_o(p^{o}_{3/2}) = k^{o}_{3/2} \qquad (4\text{-}26)$$

$$H_2(\bar{\rho}\, v^{2}_{3/2}) + \bar{\rho}\, \frac{\partial}{\partial \bar{\rho}}\, p^{2}_{3/2} = 0 \qquad (4\text{-}27a)$$

$$\frac{\partial}{\partial \bar{\rho}} (\bar{\rho}\, v^{2}_{3/2}) - 2\, \underline{w}^{2}_{3/2} = 0 \qquad (4\text{-}27b)$$

$$L_2(p^{2}_{3/2}) = 0 \qquad (4\text{-}27c)$$

$$H_2(\bar{\rho}\, \underline{v}^{2}_{3/2}) + \bar{\rho}\, \frac{\partial}{\partial \bar{\rho}}\, \underline{p}^{2}_{3/2} = \underline{g}^{2}_{3/2} \qquad (4\text{-}28a)$$

$$\frac{\partial}{\partial \bar{\rho}} (\bar{\rho}\, \underline{v}^{2}_{3/2}) + 2\, w^{2}_{3/2} = 0 \qquad (4\text{-}28b)$$

$$L_2(\underline{p}^{2}_{3/2}) = \underline{k}^{2}_{3/2} \qquad (4\text{-}28c)$$

The forcing terms are

$$h^{o}_{3/2} = - \frac{\partial}{\partial \bar{x}} (\bar{\rho}\, u^{o}_{1/2}) \qquad (4\text{-}29a)$$

$$k^{o}_{3/2} = - \frac{\partial}{\partial \bar{\rho}} \left[(\underline{w}^{1}_{1})^2 - \frac{1}{2} \bar{\rho}\, v^{1}_{1} \frac{\partial}{\partial \bar{\rho}} v^{1}_{1} - v^{1}_{1} \underline{w}^{1}_{1} \right] \qquad (4\text{-}29b)$$

$$\underline{g}^{2}_{3/2} = \frac{1}{2\bar{\rho}} (\underline{w}^{1}_{1})^2 + \frac{1}{2} v^{1}_{1} \frac{\partial}{\partial \bar{\rho}} v^{1}_{1} - \frac{1}{2\bar{\rho}} v^{1}_{1} \underline{w}^{1}_{1} \qquad (4\text{-}29c)$$

$$\underline{k}^{2}_{3/2} = - \frac{\partial}{\partial \bar{\rho}} (\bar{\rho}\, \underline{g}^{2}_{3/2}) - \left[\frac{1}{\bar{\rho}} (\underline{w}^{1}_{1})^2 - v^{1}_{1} \frac{\partial}{\partial \bar{\rho}} \underline{w}^{1}_{1} - \frac{1}{\bar{\rho}} v^{1}_{1} \underline{w}^{1}_{1} \right] \qquad (4\text{-}29d)$$

where v^{1}_{1} and $\underline{w}^{1}_{1}$ are given by equation 4-16b.

The solution of 4-25 is

$$v^{o}_{3/2} = - \frac{a}{2\pi} \frac{\sigma}{\bar{\rho}\, \bar{x}} e^{-\sigma}, \quad w^{o}_{3/2} = - \frac{m\,Re}{4\pi} \frac{\sigma}{\bar{\rho}\, \bar{x}} e^{-\sigma} \qquad (4\text{-}30)$$

where m is an arbitrary constant, related to the axial torque. $v^o_{3/2}$ and $w^o_{3/2}$ are therefore identical with the terms of equivalent order which were constructed in Chapter III (cf. equation 3-12).

Integrating 4-26, one finds

$$p^o_{3/2} = \frac{\omega^2}{4} \frac{1}{\bar{x}^2} \Big[\mathrm{Ei}(-\sigma) - \mathrm{Ei}(-2\sigma) + \frac{1}{\sigma}(e^{-\sigma} - e^{-2\sigma})$$

$$+ \frac{1}{2\sigma^2}(2e^{-\sigma} - e^{-2\sigma} - 1)\Big] - \frac{a}{4\pi}\frac{1}{\bar{x}^2} \qquad (4\text{-}31)$$

The last term on the right was obtained in Section 3. 4. Equation 4-31 shows that, if the lift is not zero, pressure does not penetrate the wake, even to the first approximation. The difference in pressure across the wake has the following expansion:

$$\bar{p}(\bar{x}, \infty) - \bar{p}(\bar{x}, 0) = \epsilon^{3/2} \frac{\omega^2}{8\bar{x}^2} (2 \log 2 - 1) + o(\epsilon^{3/2}) \qquad (4\text{-}32)$$

The term exhibited on the right balances centrifugal forces within the trailing vortices.

The solutions of 4-27 and 4-28 are

$$v^2_{3/2} = {}_h v^2_{3/2} \qquad (4\text{-}33a)$$

$$\underline{w}^2_{3/2} = h\underline{w}^2_{3/2} \qquad (4\text{-}33b)$$

$$p^2_{3/2} = 0 \qquad (4\text{-}33c)$$

$$\underline{v}^2_{3/2} = \frac{2\omega^2}{Re} \frac{1}{\bar{\rho}^3} \{ \frac{\sigma^2}{2}[Ei(-2\sigma) - Ei(-\sigma)] + \frac{\sigma}{4}[e^{-2\sigma} - 2e^{-\sigma} - e^{-\sigma}\log\sigma$$

$$+ e^{-\sigma}Ei(-\sigma)] + \frac{1}{8}[e^{-2\sigma} - 4e^{-\sigma} + 2e^{-\sigma}Ei(-\sigma) - 2e^{-\sigma}\log\sigma$$

$$- 2Ei(-2\sigma) + 2Ei(-\sigma)] + \frac{1}{4}(\frac{3}{2} + \log 2 - \gamma)(\sigma + 1)e^{-\sigma}$$

$$+ \frac{1}{4}(\log\bar{x})(\sigma + 1)e^{-\sigma} - \frac{1}{4}\log\bar{x} \} + {}_h\underline{v}^2_{3/2} \qquad (4\text{-}34a)$$

$$w^2_{3/2} = -\frac{1}{2}\frac{\partial}{\partial\bar{\rho}}(\bar{\rho}\,\underline{v}^2_{3/2}) \qquad (4\text{-}34b)$$

$$\underline{p}^2_{3/2} = \frac{\omega^2}{4}\frac{1}{\bar{x}^2}[\sigma Ei(-2\sigma) - \sigma Ei(-\sigma) + \frac{1}{2}(e^{-2\sigma} - 2e^{-\sigma}$$

$$- \frac{1}{4\sigma}(3e^{-2\sigma} - 4e^{-\sigma}) - \frac{1}{4\sigma}] \qquad (4\text{-}34c)$$

Equation 4-33c follows from matching. From 4-34 one sees that $\underline{v}^2_{3/2}$ and $w^2_{3/2}$ are associated with switchback terms $\underline{v}^2_{3/2a}$ and $w^2_{3/2a}$ (cf. Section 3.6). The switchback term in the inner expansion of the cross-flow (cf. equation 4-4b) is then

$$\vec{q}^{\,+}_{3/2a} = w^2_{3/2a}(\bar{x},\bar{\rho}) \sin 2\theta + \underline{v}^2_{3/2a}(\bar{x},\bar{\rho})\cos 2\theta \qquad (4\text{-}35)$$

4.8. <u>The Outer Term of Order ϵ^3: $\vec{q}_3$.</u>

The outer term $\vec{q}_3$ (cf. equation 4-1a) may be expressed in terms of a potential:

$$\vec{q}_3 = \tilde{\nabla}\phi_3, \qquad \tilde{\nabla}^2\phi_3 = 0 \qquad (4\text{-}36a)$$

$$\phi_3 = \phi_3^o(\tilde{r},\mu) + \phi_3^1(\tilde{r},\mu)\sin\theta + \underline{\phi}_3^1(\tilde{r},\mu)\cos\theta + \phi_3^2(\tilde{r},\mu)\sin 2\theta$$

$$+ \underline{\phi}_3^2(\tilde{r},\mu)\cos 2\theta \qquad (4\text{-}36b)$$

where the $\phi_\lambda^n(\tilde{r},\mu)$ are defined by

$$\phi_3^o(\tilde{r},\mu) = \frac{a^2}{2Re} C_1^o(\tilde{r},\mu) + \frac{1}{2}\tilde{a}_{5/2}^o \frac{\mu}{\tilde{r}^2} \qquad (4\text{-}37a)$$

$$\phi_3^1(\tilde{r},\mu) = \frac{a\omega}{2Re} C_1^1(\tilde{r},\mu) + \frac{1}{3}\tilde{a}_3^1 \frac{(1-\mu^2)^{1/2}}{\tilde{r}^2} \qquad (4\text{-}37b)$$

$$\underline{\phi}_3^1(\tilde{r},\mu) = \frac{1}{3}\underline{\tilde{a}}_3^1 \frac{(1-\mu^2)^{1/2}}{\tilde{r}^2} \qquad (4\text{-}37c)$$

$$\phi_3^2(\tilde{r},\mu) = b_{3/2}^2 B_1^2(\tilde{r},\mu) \qquad (4\text{-}37d)$$

$$\underline{\phi}_3^2(\tilde{r},\mu) = \frac{\omega^2}{2Re} C_1^2(\tilde{r},\mu) + \underline{b}_{3/2}^2 B_1^2(\tilde{r},\mu) \qquad (4\text{-}37e)$$

The functions $B_\lambda^n(\tilde{r},\mu)$ and $C_\lambda^n(\tilde{r},\mu)$ are defined in the appendix (see Section A. 3). The switchback term $\vec{q}_{3a}$ may be constructed in the usual manner, using the definition of $C_\lambda^n(\tilde{r},\mu)$ (cf. equation A-16).

The necessity of each of the terms on the right of 4-36b may be seen as follows: ϕ_3^o is the axially-symmetric potential constructed in Chapter III (see section 3. 7); the constant $\tilde{a}_1$ is renumbered in the present chapter as $\tilde{a}_{5/2}^o$. The first term on the right of 4-37b is required by the coefficient p_2^1; this coefficient has the outer expansion

$$p_2^1 \sim \epsilon^{1/2} \frac{a\omega}{Re} \frac{1}{\tilde{\rho}\tilde{x}^2} \qquad (4\text{-}38)$$

as can be verified by a direct calculation. It can also be shown that $\underline{p}_2^1$ decays exponentially, and therefore no similar term appears in 4-37c. The terms involving $\tilde{a}_3^1$ and $\underline{\tilde{a}}_3^1$ are required to match with eigensolutions in the inner expansion of pressure.* ϕ_3^2 and $\underline{\phi}_3^2$ are required to match the outer expansion of $\vec{q}^*$ with the term of order $\epsilon^{3/2}$ in the inner expansion of the cross-flow (cf. equations 4-33, 4-34).

It can be shown also that $\vec{q}_3$ is given precisely by 4-36, i.e., there are no terms of the inner expansion which match with terms of order ϵ^3 in the outer expansion other than those considered above. This can be proved by considering the terms of the inner expansion which decay algebraically; the irrotationality of the outer flow requires that the outer expansion of any such term consist of a finite sum of terms each of which is of the form $\epsilon^{\nu} f(\overline{x}) \overline{\rho}^{\pm n}(\sin n\theta, \cos n\theta)$, where f is determined by the principle of eliminability. One easily verifies that all terms of this form which are of order ϵ^3 in outer variables have been considered in the construction of $\vec{q}_3$.

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*Note that the terms of the outer expansions of velocity and pressure (equations 4-1a, 4-1b) are related by equation 3-8, Chapter III.

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APPENDIX

A. 1. The General Navier-Stokes Equations in Dimensionless Cylindrical Polar Coordinates

In dimensionless cylindrical polar coordinates (see Section 3. 1), the Navier-Stokes equations for the stationary flow of a viscous, incompressible fluid are

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial \rho^*} + \frac{w^*}{\rho^*}\frac{\partial u^*}{\partial \theta} + \frac{\partial p^*}{\partial x^*} - \frac{1}{Re}\nabla^{*2}u^* = 0 \qquad \text{(A-1a)}$$

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial \rho^*} + \frac{w^*}{\rho^*}\frac{\partial v^*}{\partial \theta} - \frac{w^{*2}}{\rho^*} + \frac{\partial p^*}{\partial \rho^*}$$

$$- \frac{1}{Re}(\nabla^{*2}v^* - \frac{v^*}{\rho^{*2}} - \frac{2}{\rho^{*2}}\frac{\partial w^*}{\partial \theta}) = 0 \qquad \text{(A-1b)}$$

$$u^* \frac{\partial w^*}{\partial x^*} + v^* \frac{\partial w^*}{\partial \rho^*} + \frac{w^*}{\rho^*}\frac{\partial w^*}{\partial \theta} + \frac{v^* w^*}{\rho^*} + \frac{1}{\rho^*}\frac{\partial p^*}{\partial \theta}$$

$$- \frac{1}{Re}(\nabla^{*2}w^* - \frac{w^*}{\rho^{*2}} + \frac{2}{\rho^{*2}}\frac{\partial v^*}{\partial \theta}) = 0 \qquad \text{(A-1c)}$$

$$\frac{\partial \rho^* u^*}{\partial u^*} + \frac{\partial \rho^* v^*}{\partial \rho^*} + \frac{\partial w^*}{\partial \theta} = 0 \qquad \text{(A-1d)}$$

where

$$\nabla^{*2} = \frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial \rho^{*2}} + \frac{1}{\rho^*}\frac{\partial}{\partial \rho^*} + \frac{1}{\rho^{*2}}\frac{\partial^2}{\partial \theta^2} \qquad \text{(A-2)}$$

A. 2. Similarity Solutions of the Equation $H_n(W) = 0$.

We seek solutions of the homogeneous equation $H_n(W) = 0$ which are of the form

$$W = \frac{\sigma^n}{\bar{\rho}^n \bar{x}^{m-n/2}} \psi(\sigma) \tag{A-3}$$

where $n = 0, 1, 2, \ldots,$ $m \geqslant 0$, and σ is defined by

$$\sigma = \frac{Re}{4} \frac{\bar{\rho}^2}{\bar{x}} = \frac{Re}{4} \frac{\rho^{*2}}{x^*} \tag{A-4}$$

By direct substitution,

$$H_n(W) = -\frac{\sigma^n}{\bar{\rho}^n \bar{x}^{m-n/2+1}} D_m^n(\psi) \tag{A-5}$$

where

$$D_m^n(\psi) = \sigma\psi'' + (\sigma+n+1)\psi' + (m + n/2)\psi \tag{A-6}$$

the prime denoting differentiation with respect to σ. Two linearly independent solutions of the equation $D_m^n(\psi) = 0$ are

$$\psi_1 = \Phi(m + n/2, n+1; -\sigma) \tag{A-7a}$$

$$\psi_2 = \psi_1 \int^{\sigma} e^{-s} s^{-(n+1)} [\psi_1(s)]^{-2} \, ds \tag{A-7b}$$

where Φ denotes the confluent hypergeometric function. Since $\Phi(a, b; 0) = 1$, one sees from A-7b that ψ_2 has a pole of order n $(n = 1, 2, \ldots)$ at the origin, and behaves as $\log \sigma$ if $n = 0$. Thus for all non-negative integer n, W is regular on the positive $\bar{x}$-axis if and only if ψ is a multiple of ψ_1.

As $\sigma \to \infty$, we have (cf. reference 6, p. 265)

$$\psi_1 \sim \frac{\Gamma(n+1)}{\Gamma(1-m+n/2)}\,\sigma^{-(m+n/2)}\,{}_2F_0(m+n/2,\, m-n/2;\sigma^{-1}) \tag{A-8}$$

One sees from this expansion that ψ_1 decays exponentially if and only if

$$m - n/2 = 1, 2, \ldots \tag{A-9}$$

If $2m = n$, A-8 may be replaced by the expansion

$$\psi_1 \sim \frac{n!}{\sigma^n} + \ldots \tag{A-10}$$

where the dots indicate a transcendentally small remainder. The right-hand side of A-10 is replaced by $\log \sigma$ for the case $n = 0$. Comparing A-3 and A-10, one sees that there exist solutions of $H_n(W) = 0$ whose algebraically decaying part is a function of $\bar{\rho}$ alone; for these solutions $2m = n = 0, 1, 2, \ldots$. These exceptional solutions appear as eigensolutions in the inner expansion of the transverse velocity components. The case $m = n = 0$ is then eliminated by matching.

The Eigensolutions W_m^n

The solutions W of interest in Chapter III and IV are regular on the positive $\bar{x}$-axis and satisfy homogeneous matching conditions at $\bar{\rho} = \infty$. We define

$$W_m^n = \frac{\sigma^n}{\bar{\rho}^n \bar{x}^{m-n/2}}\,\Phi(m+n/2,\, m+1;-\sigma) \tag{A-11}$$

where $m - n/2 = 1, 2, \ldots$, $n = 0, 1, \ldots$, or $2m = n = 1, 2, \ldots$. The desired eigensolutions are constant multiples of the solutions W_m^n.

A. 3. Several Solutions of Laplace's Equation

We consider first solutions of Laplace's equation which are of the form

$$R_\lambda^n(\tilde{r},\mu,\theta) = B_\lambda^n(\tilde{r},\mu)(\sin n\theta, \cos n\theta) = \tilde{r}^{-(\lambda+1)} G_\lambda^n(\mu)(\sin n\theta, \cos n\theta),$$

$$n = 0, 1, 2, \ldots \qquad \text{(A-12)}$$

where

$$\tilde{r} = (\tilde{x}^2 + \tilde{\rho}^2)^{1/2}, \quad \mu = \frac{\tilde{x}}{\tilde{r}} = \frac{x^*}{r^*} \qquad \text{(A-13)}$$

By direct substitution, G_λ^n is a solution of Legendre's differential equation. If G_λ^n is required to be a regular function of μ on the interval $-1 \leq \mu \leq +1$, then $\lambda = 0, 1, 2, \ldots$ and $G_\lambda^n = P_\lambda^n$, the associated Legendre functions of the first kind. Under these conditions R_λ^n is within a multiplicative constant, the potential of a term in the outer expansion of $-\vec{q}^* \cdot i$ which matches with the eigensolutions of order $\nu = \lambda + n/2 + 3/2$ in the inner expansion of $\overline{p}$. Since $P_\lambda^n = 0$ whenever $n - \lambda = 1, 2, \ldots$, we have, by matching,

$$\tilde{a}_\nu^n = \underline{\tilde{a}}_\nu^n = 0, \; (3/2\, n - \nu + 1/2 = 0, 1, 2, \ldots) \qquad \text{(A-14)}$$

in the eigensolution 4-13.

In order to match the inner and outer expansions of v^* and w^* for the general three-dimensional case, additional solutions of the form A-12 are required for which $\lambda = n - 1$. For details concerning the solutions of Legendre's equation applicable to these cases, see reference 7. For our purposes, we define

$$G_{n-1}^{n}(\mu) = \frac{\Gamma(2n-1)}{2^{2n-1}[\Gamma(n)]^2} \frac{(1+\mu)^{n/2}}{(1-\mu)^{1-n/2}} F(1, 1-n, 2-2n; \frac{2}{1-\mu}) \quad \text{(A-15)}$$

where F denotes the hypergeometric function. The function cR_{n-1}^{n} (c = arbitrary constant) is the potential of a term in the outer expansion of $\vec{q}^{*}$ which matches with the eigensolutions $cW_{n/2}^{n}(\sin n\theta, \cos n\theta)$ appearing in the inner expansion of v^{*} and w^{*}.

A final class of harmonic functions which are required in the construction are defined as follows:

$$S_{\lambda}^{n}(\tilde{r}, \mu, \theta) = C_{\lambda}^{n}(\tilde{r}, \mu)(\sin n\theta, \cos n\theta) = \tilde{r}^{-(\lambda+1)}[K_{\lambda}^{n}(\mu)$$

$$- G_{\lambda}^{n}(\mu)\log \tilde{r}](\sin n\theta, \cos n\theta) \quad \text{(A-16)}$$

where

$$G_1^{o}(\mu) = \mu \quad \text{(A-17a)}$$

$$K_1^{o}(\mu) = \mu \log(1-\mu) + 1 \quad \text{(A-17b)}$$

$$G_o^{1}(\mu) = -\frac{1}{2}\left(\frac{1+\mu}{1-\mu}\right)^{1/2} \quad \text{(A-17c)}$$

$$K_o^{1}(\mu) = \frac{1}{2}\left(\frac{1-\mu}{1+\mu}\right)^{1/2} \log\left(\frac{1-\mu}{2}\right) \quad \text{(A-17d)}$$

$$G_1^{1}(\mu) = -(1-\mu^2)^{1/2} \quad \text{(A-17e)}$$

$$K_1^{1}(\mu) = \mu\left(\frac{1+\mu}{1-\mu}\right)^{1/2} - \mu(1-\mu^2)^{1/2}\log(1-\mu) \quad \text{(A-17f)}$$

$$G_1^{2}(\mu) = -\frac{1}{4}\left(\frac{\mu^2-\mu-2}{1-\mu}\right) \quad \text{(A-17g)}$$

$$K_1^{2}(\mu) = \frac{1}{4}\left[1 + 2\mu + \left(\frac{\mu^2+\mu-2}{1+\mu}\right)\log\left(\frac{1-\mu}{2}\right)\right] \quad \text{(A-17h)}$$

The solutions S_{λ}^{n} are regular everywhere except at the origin and along the postive x-axis. Note also that these solutions always require switchback terms (see Section 3. 6).

A. 4. The Calculation of Force and Moment

The expansions given above (cf. equations 4-1 and 4-4) allow a determination of the force and moment experienced by a solid (or a closed streamsurface) in terms of the arbitrary constants. We define the dimensionless tensor $\underline{\underline{A}}^*$ by

$$\underline{\underline{A}}^* = \vec{q}^* \circ \vec{q}^* + p^* \underline{\underline{I}} - \frac{1}{Re} \operatorname{def}^* \vec{q}^* \qquad \text{(A-18)}$$

where $\underline{\underline{I}}$ = identity tensor and $\operatorname{def}^* \vec{q}^*$ = dimensionless deformation tensor. The conservation laws for moment and angular momentum are respectively

$$\nabla^* \cdot \underline{\underline{A}}^* = 0 \qquad \text{(A-19a)}$$

$$\nabla^* \cdot \underline{\underline{M}}^* = 0, \quad \underline{\underline{M}}^* = \vec{r}^* \times \underline{\underline{A}}^* \qquad \text{(A-19b)}$$

If Gauss' Theorem is applied to A-19, the surface integral will consist of two parts: the first may be taken to be the body surface S_o; the second may be chosen as a sphere of arbitrarily large radius centered at the origin. The asymptotic expansion may be used to evaluate the integral over the latter. One finds

$$\oiint_{S_o} \underline{\underline{A}}^* \cdot d\vec{s}^* = a\,\vec{i} + b\,\vec{j} \qquad \text{(A-20a)}$$

$$\oiint_{S_o} \underline{\underline{M}}^* \cdot d\vec{s}^* = m\,\vec{i} - \left(\frac{4}{3}\pi \underline{\tilde{a}}_3^1 + \frac{4\pi}{Re} \underline{a}_1^1\right)\vec{j}$$
$$- \left[\frac{4}{3}\pi \tilde{a}_3^1 + \frac{4\pi}{Re} a_1^1 + \frac{2\pi a\omega}{Re}(2 \log 2 - \gamma)\right]\vec{k} \qquad \text{(A-20b)}$$

where $d\vec{s}^*$ is the dimensionless area element, directed toward the interior of the body, and γ = Euler's Constant. The terms on the left of A-20 are respectively the dimensionless force and moment experienced on the body. The term on the right of A-20b, involving $\alpha\omega$, is the contribution to moment caused by "deflection of the wake." This contribution vanishes when either the lift or the drag on the body is zero.